# Vector Strategies

## Index

**Projection of a point P on a line**

- the point $P'$ where $PP'$ is the shortest distance from point P to $P'$ on the line.

**Projection of a line segment on a line**

The projection of a line segment denoted by ***b*** on a line with direction vector ***a***

- magnitude is $|\boldsymbol{b}|\cos\theta$ where θ is the angle between the two line segments.

**Distance from a point to a line**

- Consider a point P lying outwith the plane. Let $P'$ be the projection of P on the line $L$.
- Let the line $L$ have parameter $t$ and direction vector ***u***.
- Since $P'$ lies on $L$ its coordinates can be expressed in terms of $t$.
- The components of $PP'$ can be expressed in terms of t.
- Since $PP'$ and ***u*** are perpendicular, their scalar product is zero.
- Solving this equation for $t$ gives the coordinates of $P'$
- The length of $PP'$ can be found from the distance formula.

**Example:** Find the distance from the point P(15, –9, –2) to the line $L$

with equations: $\frac{x-35}{8} = \frac{y+43}{-10} = \frac{z-62}{13} = t$

**Solution:**

Let $P'$ be the projection of P on the line $L$.

Then the coordinates of $P'$ will be $(8t+35, -10t-43, 13t+62)$ for some value of $t$.

( since for the line $L$, $x - 35 = 8t$, $y + 43 = -10t$, $z - 62 = 13t$ )

$$\text{Hence, } PP' = \begin{pmatrix} 8t+35 \\ -10t-43 \\ 13t+62 \end{pmatrix} - \begin{pmatrix} 15 \\ -9 \\ -2 \end{pmatrix} = \begin{pmatrix} 8t+20 \\ -10t-34 \\ 13t+64 \end{pmatrix} \text{ and } \boldsymbol{u} = \begin{pmatrix} 8 \\ -10 \\ 13 \end{pmatrix}$$

Since $PP'$ and ***u*** are perpendicular, then $PP' \cdot \boldsymbol{u} = 0$

$$\text{Then: } \begin{pmatrix} 8t+20 \\ -10t-34 \\ 13t+64 \end{pmatrix} \cdot \begin{pmatrix} 8 \\ -10 \\ 13 \end{pmatrix} = 0 \quad \text{i.e.} \quad 64t + 160 + 100t + 340 + 169t + 832 = 0$$

So, $333t + 1332 = 0 \Rightarrow t = -4$

$$\text{Thus } PP' = \begin{pmatrix} 8t+20 \\ -10t-34 \\ 13t+64 \end{pmatrix} = \begin{pmatrix} -32+20 \\ 40-34 \\ -52+64 \end{pmatrix} = \begin{pmatrix} -12 \\ 6 \\ 12 \end{pmatrix} = 6\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

Hence the distance from P to $P'$ is $|PP'| = 6\sqrt{(-2)^2 + 1^2 + 2^2} = 6\sqrt{9} = 18$ units.

**Distance between parallel planes**

Given two planes $ax + by + cz = m$ and $ax + by + cz = n$

- ensure the planes are parallel by checking that the normal is the same (note one may be a scalar multiple of the other)
- express in the form $\boldsymbol{a.x} = m$ and $\boldsymbol{a.x} = n$ for both planes
- the distance between the planes is $\dfrac{|m-n|}{|\boldsymbol{a}|}$

**Example:** Find the distance between the parallel planes with equations $x + y + z = 2$ and $2x + 2y + 2z + 3 = 0$

**Solution:** Ensure normal is same for both planes by writing in form $\boldsymbol{a.x} = m$ and $\boldsymbol{a.x} = n$

$x + y + z = 2$ and $x + y + z = -\dfrac{3}{2}$

Hence normal is $\begin{pmatrix}1\\1\\1\end{pmatrix}$ and $m = 2$ and $n = -\dfrac{3}{2}$

Use the formula $d = \dfrac{|m-n|}{|\boldsymbol{a}|}$

$$\Rightarrow d = \frac{\left|2-\left(-\frac{3}{2}\right)\right|}{\sqrt{1^2+1^2+1^2}} = \frac{\frac{7}{2}}{\sqrt{3}} = \frac{7}{2\sqrt{3}} = \frac{7\sqrt{3}}{6}$$

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**Distance from a point to a plane**

- Consider a point P lying outwith the plane. Let $P'$ be the projection of P on the plane.
- Then $PP'$ is parallel to the normal vector to the plane, which can be extracted from the equation of the plane.
- Using the normal vector and the coordinates of the point P, write down the equation of the line $PP'$
- $P'$ is the intersection of the line $PP'$ with the plane. Its coordinates can be found by finding the value of the parameter *t* and calculating (*x, y, z*) for $P'$.
- The length of $PP'$ can be found from the distance formula.

**Example:** Find the distance of the point P(1, 2, 1) from the plane with equation $x - y + 2z = 5$

**Solution:**

A normal vector to the plane is: $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ from equation of plane.

Let $P'$ be the projection of P on the plane.

The equation of $PP'$ $\frac{x-1}{1} = \frac{y-2}{-1} = \frac{z-1}{2} = t$ from normal $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and point P(1, 2, 1).

Any point on this line is of form: $(t+1,\ -t+2,\ 2t+1)$ from equation of line.

This form will also satisfy the equation of the plane, so $(t+1) - (2-t) + 2(2t+1) = 5$

We can solve this and find the value for *t*: $\Rightarrow\ t = \frac{2}{3}$. Hence $P'\,is\ \left(\frac{5}{3}, \frac{4}{3}, \frac{7}{3}\right)$

$$PP' = \begin{pmatrix} \frac{5}{3} - 1 \\ \frac{4}{3} - 2 \\ \frac{7}{3} - 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{4}{3} \end{pmatrix} = \frac{2}{3}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

and hence using distance formula: $PP' = \frac{2}{3}\sqrt{1^2 - (-1)^2 + 2^2} = \frac{2}{3}\sqrt{1+1+4} = \frac{2}{3}\sqrt{6}$

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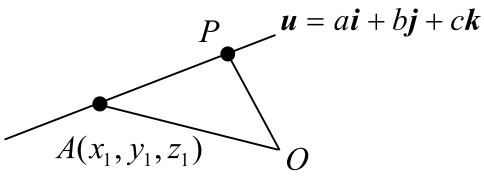
**Equation of a line**

A line is completely determined when we know its direction vector and a point on the line.

Consider the line L which passes through point $A(x_1, y_1, z_1)$ with direction vector $u = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$ and let $P$ be any point on the line.




Then $\overrightarrow{AP} = t\boldsymbol{u}$ for some scalar $t$.

And since $\overrightarrow{AP} = \boldsymbol{p} - \boldsymbol{a}$ then $\boldsymbol{p} - \boldsymbol{a} = t\boldsymbol{u}$ thus $\boldsymbol{p} = \boldsymbol{a} + t\boldsymbol{u}$

In component form this becomes: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + t\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x_1 + at \\ y_1 + bt \\ z_1 + ct \end{pmatrix}$

In parametric form this is: $x = x_1 + at, \quad y = y_1 + bt, \quad z = z_1 + ct$

In symmetric form this is: $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c} = t$

**Example**: Write down the symmetric form of the equations of the line which passes through (1, −2, 8) and is parallel to $3\boldsymbol{i} + 5\boldsymbol{j} + 11\boldsymbol{k}$.

**Solution:** Using the point and the direction vector directly,

$$\frac{x-1}{3} = \frac{y+2}{5} = \frac{z-8}{11} = t$$

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**To check if a point lies on a line**

- Obtain the parametric form of the equation
- Substitute the x, y and z coordinates of the point in the equations and verify that t is the same for all three equations

**Example**: Find out if the point (-2, -7, -3) lies on the line $\dfrac{x-1}{3} = \dfrac{y+2}{5} = \dfrac{z-8}{11} = t$

**Solution:** Write down the parametric equations:

$x = 3t + 1, \quad y = 5t - 2, \quad z = 11t + 8$

Substitute for the point to find t for each equation.

$-2 = 3t + 1, \quad -7 = 5t - 2, \quad -3 = 11t + 8$

$t = -1, \qquad t = -1 \qquad t = -1$

Since the results are consistent, the point lies on the line.

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**Equation of a line from two points**

- Obtain the direction vector from the two points
- Use either of the given points to substitute into the symmetric form.

**Example**: Find the equations of the line passing through A(2, 1, 3) and B(3, 4, 5)

**Solution:** The line is parallel to AB, so direction vector is: $\begin{pmatrix} 3-2 \\ 4-1 \\ 5-3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$

Choose a point, say A(2, 1, 3) and substitute into symmetric form:

$$\frac{x-2}{1} = \frac{y-1}{3} = \frac{z-3}{2} = t$$

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**Vector equation of a line from two points**

The vector equation of a line through point A with direction vector $\boldsymbol{u}$ is: $\boldsymbol{p} = \boldsymbol{a} + t\boldsymbol{u}$

When using vector equations we usually replace $\boldsymbol{p}$ with $\boldsymbol{r}$.

If we have two points A and B, then the direction vector is $\boldsymbol{u} = \boldsymbol{b} - \boldsymbol{a}$

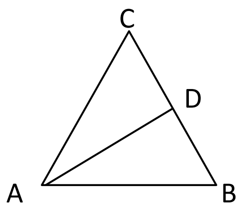
And so we get: $\boldsymbol{r} = \boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a})$ which can be rearranged as follows:

$\Rightarrow \quad \boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b} - t\boldsymbol{a} \quad \Rightarrow \quad \boldsymbol{r} = (1-t)\boldsymbol{a} + t\boldsymbol{b}$

**Example:** Find in terms of ***a, b*** and ***c*** the vector equation of the median AD of $\Delta$ABC.

**Solution:**

Draw a diagram




$\boldsymbol{r} = \boldsymbol{a} + t\left(\overrightarrow{AD}\right)$

Express $\overrightarrow{AD}$ *as* $\boldsymbol{d} - \boldsymbol{a}$ $\quad \Rightarrow \quad \boldsymbol{r} = \boldsymbol{a} + t(\boldsymbol{d} - \boldsymbol{a})$

Take out a factor of $\boldsymbol{a}$: $\quad \Rightarrow \quad \boldsymbol{r} = (1-t)\boldsymbol{a} + t\boldsymbol{d}$

Replace $\boldsymbol{d}$ with mid-point $\quad \Rightarrow \quad \boldsymbol{r} = (1-t)\boldsymbol{a} + t\left[\frac{1}{2}(\boldsymbol{b} + \boldsymbol{c})\right]$

Simplify: $\quad \Rightarrow \quad \boldsymbol{r} = (1-t)\boldsymbol{a} + \frac{1}{2}t\boldsymbol{b} + \frac{1}{2}t\boldsymbol{c}$

Replace parameter $t$ with $2k$ to simplify fractions. $\quad \Rightarrow \quad \boldsymbol{r} = (1-2k)\boldsymbol{a} + k\boldsymbol{b} + k\boldsymbol{c}$

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**Equation of a plane from a given point P and a normal $\boldsymbol{a}$**

- find $k$ using $\boldsymbol{a.p} = k$
- equation is $ax + by + cz = k$ where $\boldsymbol{a} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and a general point is $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

**Example:** P and Q are the points (1, 2, 3) and (2, 1 −4) respectively.
Find the equation of the plane perpendicular to PQ which contains the point P.

**Solution:** $\overrightarrow{PQ} = \boldsymbol{q} - \boldsymbol{p} = \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -7 \end{pmatrix}$

$\overrightarrow{PQ}$ is therefore the normal to the plane and P (1, 2, 3) lies on the plane.

Hence: $\boldsymbol{a.p} = k \Rightarrow \begin{pmatrix} 1 \\ -1 \\ -7 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 1 - 2 - 21 = -22$

and so the equation of the plane is: $x - y - 7z = -22$

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**Equation of a plane from three points P, Q, R on the plane**

- find two vectors that lie in the plane $\overrightarrow{PQ}, \overrightarrow{PR}$ (note order of points)
- find the normal using vector product: $\overrightarrow{PQ} \times \overrightarrow{PR}$
- choose any of the points and take scalar product with the normal to find $k$
- equation: $ax + by + cz = k$ where $\boldsymbol{a} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ (normal) and $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ (general point)

**Example:** Find the equation of the plane which passes through the points
A(−2, 1, 2), B(0, 2, 5) and C(2, −1, 3)

**Solution:** $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a} = \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ *and* $\overrightarrow{AC} = \boldsymbol{c} - \boldsymbol{a} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$

Normal is $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 1 & 3 \\ 4 & -2 & 1 \end{vmatrix} = \begin{pmatrix} 7 \\ 10 \\ -8 \end{pmatrix}$

Using point A to obtain $k$, we get: $k = \begin{pmatrix} 7 \\ 10 \\ -8 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} = -20$

So equation of plane is: $7x + 10y - 8z = -20$

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**Vector equation of a plane (parametric form)**

A plane can be defined by any two non-zero non-parallel vectors which lie upon it.

Let R be any point on the plane and A, B, C are three known points with position vectors ***r, a, b, c*** respectively then $\boldsymbol{r} = (1-t-u)\boldsymbol{a} + t\boldsymbol{b} + u\boldsymbol{c}$

an alternative form is: $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b} + u\boldsymbol{c}$ where A is a point on the plane and ***b*** and ***c*** are vectors parallel to the plane

**Given three points on the plane** – to find the vector equation of the plane

- substitute the position vectors into the above equation

**Example:** Find the equation of the plane in vector form, which contains the points: A(1, 2, –1), B(–2, 3, 2) and C(4, 5, 2).

**Solution:** $\boldsymbol{a} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix}$, $\boldsymbol{c} = \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix}$ from the position vectors of the points

Use the equation: $\boldsymbol{r} = (1-t-u)\boldsymbol{a} + t\boldsymbol{b} + u\boldsymbol{c}$ and substitute

$$\boldsymbol{r} = (1-t-u)\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + t\begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix} + u\begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 1-t-u-2t+4u \\ 2-2t-2u+3t+5u \\ -1+t+u+2t+2u \end{pmatrix} = \begin{pmatrix} 1-3t+3u \\ 2+t+3u \\ -1+3t+3u \end{pmatrix}$$

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**Find the point on a plane given in parametric vector form for values of parameters *t* and *u*.**

- Substitute the values of the parameters into the parametric equation

**Example:** Find a point on the plane $\boldsymbol{r} = \begin{pmatrix} 1-3t+3u \\ 2+t+3u \\ -1+3t+3u \end{pmatrix}$ corresponding to the parameter values $t = 2$, $u = 3$

**Solution:**

Substitute the parameter values $t = 2$, $u = 3$

$$\boldsymbol{r} = \begin{pmatrix} 1-3(2)+3(3) \\ 2+(2)+3(3) \\ -1+3(2)+3(3) \end{pmatrix} = \begin{pmatrix} 4 \\ 13 \\ 14 \end{pmatrix}$$ and this is the position vector of the point.

So the point is: P(4, 13, 14)

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**To check a point lies on a plane with the given vector equation**

- equate *x, y, z* components of the point with the parametric components of the equation
- solve the three simultaneous equations and check the solutions are consistent.

**Example:** Find condition such that P(−8, 9, 14) lies on

$$\boldsymbol{r} = (1-3t+3u)\boldsymbol{i} + (2+t+3u)\boldsymbol{j} + (-1+3t+3u)\boldsymbol{k}$$

**Solution:**

Equate the *x, y, z* components:

$-8 = 1-3t+3u$ (1)

$9 = 2+t+3u$ (2)

$14 = -1+3t+3u$ (3)

Solve equations simultaneously (2) – (1) $\Rightarrow$ $t = 4,\ u = 1$

Check for consistency by substituting into (3)
i.e. 14 = −1 + 12 + 3 = 14. Thus solutions are consistent.
So for P to be a point on the plane $\Rightarrow$ $t = 4,\ u = 1$

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**To find the cartesian equation of a plane from the parametric vector form**

- Rearrange into vector form $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b} + u\boldsymbol{c}$
- compare components to identify point $\boldsymbol{a}$ and the vectors $\boldsymbol{b}$ and $\boldsymbol{c}$
- Find the vector normal to the plane i.e. $\boldsymbol{b}\times\boldsymbol{c}$
- Find *k* from $\boldsymbol{a}.(\boldsymbol{b}\times\boldsymbol{c}) = k$
- Substitute into cartesian form $\boldsymbol{a.n} = k$ where $\boldsymbol{n}$ is the normal

**Example:** Find the cartesian equation of the plane whose vector equation is

$$\boldsymbol{r} = (2+t+3u)\boldsymbol{i} + (1+t-3u)\boldsymbol{j} + (3-t-7u)\boldsymbol{k}$$

**Solution:** multiply out the brackets and regroup in form $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b} + u\boldsymbol{c}$

$\boldsymbol{r} = 2\boldsymbol{i} + t\boldsymbol{i} + 3u\boldsymbol{i} + \boldsymbol{j} + t\boldsymbol{j} - 3u\boldsymbol{j} + 3\boldsymbol{k} - t\boldsymbol{k} - 7u\boldsymbol{k}$

$\Rightarrow\ \boldsymbol{r} = (2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}) + t(\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}) + u(3\boldsymbol{i} - 3\boldsymbol{j} - 7\boldsymbol{k})$

Hence: $\Rightarrow\ a = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}\ b = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\ c = \begin{pmatrix} 3 \\ -3 \\ -7 \end{pmatrix}$

Find normal to the plane $\boldsymbol{b}\times\boldsymbol{c} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 1 & -1 \\ 3 & -3 & -7 \end{vmatrix} = \begin{pmatrix} -10 \\ 4 \\ -6 \end{pmatrix}$

Find $k = \boldsymbol{a.n} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} -10 \\ 4 \\ -6 \end{pmatrix} = -34$

Hence equation of plane is: $-10x + 4y - 6z = -34$

simplifying (divide by −2 ): $5x - 2y + 3z = 17$

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**Find three points on a plane from the parametric vector form**

- Rearrange into vector form $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b} + u\boldsymbol{c}$
- compare components to identify point $\boldsymbol{a}$ and the vectors $\boldsymbol{b}$ and $\boldsymbol{c}$
- Find the vector normal to the plane i.e. $\boldsymbol{b} \times \boldsymbol{c}$
- Find $k$ from $\boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c}) = k$
- Substitute into cartesian form $\boldsymbol{a}.\boldsymbol{n} = k$ where $\boldsymbol{n}$ is the normal

**Example:** Find three points on the plane whose vector equation is

$$\boldsymbol{r} = (2 + t + 3u)\boldsymbol{i} + (1 + t - 3u)\boldsymbol{j} + (3 - t - 7u)\boldsymbol{k}$$

**Solution:** multiply out the brackets and regroup in form $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b} + u\boldsymbol{c}$

$$\boldsymbol{r} = 2\boldsymbol{i} + t\boldsymbol{i} + 3u\boldsymbol{i} + \boldsymbol{j} + t\boldsymbol{j} - 3u\boldsymbol{j} + 3\boldsymbol{k} - t\boldsymbol{k} - 7u\boldsymbol{k}$$

$$\Rightarrow \boldsymbol{r} = (2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}) + t(\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}) + u(3\boldsymbol{i} - 3\boldsymbol{j} - 7\boldsymbol{k})$$

Hence: $\Rightarrow \boldsymbol{r} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + u\begin{pmatrix} 3 \\ -3 \\ -7 \end{pmatrix}$

Rearrange to the form $\boldsymbol{r} = (1 - t - u)\boldsymbol{a} + t\boldsymbol{b} + u\boldsymbol{c}$

$\Rightarrow \boldsymbol{r} = (1 - t - u)\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + t\left[\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right] + u\left[\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ -3 \\ -7 \end{pmatrix}\right]$ by rearrangement.,

Simplify: $\Rightarrow \boldsymbol{r} = (1 - t - u)\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} + u\begin{pmatrix} 5 \\ -2 \\ -4 \end{pmatrix}$

Hence the three points are: (2, 1, 3), (3, 2, 2) and (5, –2, –4)

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**Normal of a plane from its equation**

from the equation: $ax + by + cz = k$

- the normal is $\boldsymbol{a} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

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**Check if a given point lies on a given plane**

- Check if the coordinates of the point satisfy the equation of the plane.

**Example:** Does the point A(2, 1, 3) lie on the plane $5x-2y+3z=17$

**Solution:** Put the point into the equation:

$5(2)-2(1)+3(3)=10-2+9=17$

Point satisfies equation, so the point lies on the plane.

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**Angle between two vectors**

if the two vectors are ***a*** and ***b*** then the angle between the vectors, θ, is given by

- $\cos\theta=\dfrac{\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$. If $\cos\theta<0$ then θ is obtuse.

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**Angle between two planes**

Given two planes: $ax+by+cz=k$ and $dx+ey+fz=h$

- obtain the normal vector to each plane $\boldsymbol{a}=\begin{pmatrix}a\\b\\c\end{pmatrix}$ and $\boldsymbol{b}=\begin{pmatrix}d\\e\\f\end{pmatrix}$

- the angle between the planes is the angle between the normals $\cos\theta=\dfrac{\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$

**Example:** Find the angle between the planes with equations

$x+2y+z=5$ and $x+y=0$.

**Solution:** By inspection, the normals to the planes are: $\begin{pmatrix}1\\2\\1\end{pmatrix}$ and $\begin{pmatrix}1\\1\\0\end{pmatrix}$

Find the angle between using: $\cos\theta=\dfrac{\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$

Hence: $\cos\theta=\dfrac{1+2+0}{\sqrt{6}\sqrt{2}}=\dfrac{3}{\sqrt{12}}=\dfrac{3}{2\sqrt{3}}=\dfrac{\sqrt{3}}{2}$

and so $\theta=30°$

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**Angle between two lines**

- Put the two lines in symmetric form
- Obtain the direction vector for each line
- The angle between the two lines is the angle between their two direction vectors.
- Use $\cos\theta = \dfrac{\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$

**Example:** Find the acute angle between the lines with equations

$$\frac{x-2}{1}=\frac{y+1}{-2}=\frac{z-11}{-1} \quad \text{and} \quad x=-t+3, \quad y=t-4, \quad z=-8$$

**Solution:**

Put the second equation into symmetric form: $\dfrac{x-3}{-1} = \dfrac{y+4}{1} = \dfrac{z+8}{0}$

Obtain the direction vectors for both lines: $\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$ *and* $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

The angle between the two lines is the angle between their direction vectors:

$$\cos\theta = \frac{-1-2+0}{\sqrt{6\times 2}} = -\frac{\sqrt{3}}{2} \quad \Rightarrow \quad \theta = 150°$$

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**Angle between a line and a plane**

The angle between a line and a plane is the complement of the angle between the line and the normal to the plane.

- Obtain the direction vector of the line
- Obtain the direction vector of the plane
- Use: $\sin\theta = \dfrac{|\boldsymbol{a}.\boldsymbol{b}|}{|\boldsymbol{a}||\boldsymbol{b}|}$ use modulus of the scalar product because angle is less than 90°

**Example:** Find the angle between the line $\dfrac{x-7}{3}=\dfrac{y-11}{4}=\dfrac{z-24}{13}$ and the plane $6x+4y-5z=28$

**Solution:**

Direction vector of line is: $3\boldsymbol{i}+4\boldsymbol{j}+13\boldsymbol{k} = \begin{pmatrix} 3 \\ 4 \\ 13 \end{pmatrix}$

Direction vector of plane is: $6\boldsymbol{i}+4\boldsymbol{j}-5\boldsymbol{k} = \begin{pmatrix} 6 \\ 4 \\ -5 \end{pmatrix}$

$$\sin\theta = \frac{|18+16-65|}{\sqrt{194\times 77}} = \frac{31}{\sqrt{194\times 77}} \quad \Rightarrow \quad \theta = \sin^{-1}\left(\frac{31}{\sqrt{194\times 77}}\right) \quad \Rightarrow \quad \theta = 14.7°$$

**Intersection of a line and a plane**

- Change the equation of the line from symmetric to parametric form if necessary.
- Substitute x, y and z into the equation of the plane to give an equation in t.
- Solve the equation to find t.
- Substitute back into the parametric form of the line to obtain the point of intersection.

**Example:** Given the line $\frac{x-7}{3}=\frac{y-11}{4}=\frac{z-24}{13}$ and the plane $6x+4y-5z=28$.

Find the point of intersection.

**Solution:**

Putting the line into parametric form: $x=3t+7,\ y=4t+11,\ z=13t+24$

Substitute into equation of plane: $6(3t+7)+4(4t+11)-5(13t+24)=28$

Simplify: $18t+42\ +16t+44\ -65t-120=28$

$\Rightarrow\ -31t-34=28\ \Rightarrow\ -31t=62\ \Rightarrow\ t=-2$

Substituting for parameters in: $x=3t+7,\ y=4t+11,\ z=13t+24$

The point of intersection is: $(1,\ 3,\ -2)$

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**Intersection of two lines**

Two lines in space can be: (i) parallel, (ii) intersect at a point, (ii) skew (no intersection)

- Express the lines in parametric form using parameters $t_1$ and $t_2$
- Equate corresponding expressions for x, y, z producing three equations in two unknowns.
- Use two of the equations to find the values for $t_1$ and $t_2$
- Substitute for the parameters $t_1$ and $t_2$ in the third equation.
- If they satisfy the third equation , then the point of intersection has been found, if they don't then the lines do not intersect.

**Example:** Show that the lines with equations: $x-5=-(y+2)=z$ and

$\frac{x-12}{5}=\frac{y+3}{-2}=\frac{z-5}{4}$ intersect, and find the point of intersection.

**Solution:**

Using the parameters $t_1$ and $t_2$ we get

$x=t_1+5,\ y=-t_1-2,\ z=t_1$ and $x=5t_2+12,\ y=-2t_2-3,\ z=4t_2+5$

Equate corresponding coordinates:

$t_1=5t_2+7$ (1)

$t_1=2t_2+1$ (2)

$t_1=4t_2+5$ (3)

Subtracting (2) from (1) we find: $3t_2+6=0\ \Rightarrow\ t_2=-2$ and hence $t_1=-3$

By inspection we see that these values satisy (3): $-3=4(-2)+5$

Thus the lines intersect at a point: $x=t_1+5,\ y=-t_1-2,\ z=t_1$ with $t_1=-3$

or $x=5t_2+12,\ y=-2t_2-3,\ z=4t_2+5$ with $t_2=-2$

The point of intersection is therefore: $(2,1,-3)$

**Intersection of two planes**

Two planes must either be parallel or intersect in a line.
To determine the equation of the line of intersection, we need to know its direction vector and a point on the line.

To find a point on the line:

- The line must cross the (x, y) plane, which has equation z = 0 or be parallel to it.
- If it crosses it, set z = 0 in the equations of both planes to obtain a pair of simultaneous equations in x and y.
- Solving these two equations will provide a point on the line $(x_1, y_1, 0)$
- If the line is parallel to the (x, y) plane then a similar point can be found on the (x, z) plane.

To find the direction vector:

- The line of intersection lies in both planes
- Its direction vector is perpendicular to the normal vector of each plane.
- The direction vector is parallel to the vector product of these normal vectors.

**Example:** Find the equations of the line of intersection of the planes with equations:

$$x - 2y + 3z = 1 \quad \text{and} \quad 2x + y + z = -3$$

**Solution:**

Let z = 0, then $x - 2y = 1$ and $2x + y = -3$

$\Rightarrow\ 2(1 + 2y) + y = -3$ (substituting first equation into second.)

$\Rightarrow\ 5y = -5 \ \Rightarrow\ y = -1 \ \Rightarrow x = -1$

$\Rightarrow\ (-1, -1, 0)$ lies on the line of intersection.

Normal vectors are: $\boldsymbol{u} = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{v} = 2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & -2 & 3 \\ 2 & 1 & 1 \end{vmatrix} = \begin{pmatrix} -5 \\ 5 \\ 5 \end{pmatrix} = -5\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

Hence equations of the line of intersection are: $\dfrac{x+1}{1} = \dfrac{y+1}{-1} = \dfrac{z}{-1}$

**Intersection of three planes**

There are 6 possible cases for these intersections

- Start off by solving the three equations of the planes simultaneously using Gaussian elimination or augmented matrix form and reduce to upper triangular form where possible using EROs.
- **A unique solution** to these 3 equations gives a single point of intersection.
- **A single redundancy** indicates a line of intersection
  find the equation of the line in terms of the parameter $t$.
  i.e. start off with $z = t$ and then find expressions for $x$ and $y$,
  then you can write in the symmetric form.
- **Two redundancies** indicate a plane of intersection.
  The three planes coincide and will be identical or scalar multiples of each other.
  Choose any of the planes as the equation of the plane of intersection.
- **One inconsistency** indicates 2 or 3 lines of intersection.
  If **two** of the planes are parallel, then there will be 2 lines of intersection.
  Identify the two parallel planes (scalar multiples)
  Solve the equation of the non-parallel plane with each of the other two planes simultaneously, in turn, and use the parameter t to obtain two equations for the lines of intersection and use back substitution.
  If **none** of the planes are parallel, then solve each pair simultaneously
  Solve the equation of the each pair of planes simultaneously, in turn, and use the parameter t to obtain two equations for the lines of intersection and use back substitution.
- **Two inconsistencies** indicate that the planes do not intersect and that they are all parallel.